CONTROLLABILITY AND
OBSERVABILITY OF
MATRIX SYLVESTER SYSTEMS ON
TIMESCALES

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Abstract
This paper presents several fundamental results concerning the controllability and observability criteria for \( \Delta \)-differential matrix Sylvester system

\[
X'(t) = A(t)X(t) + X(t)B(t) + F_1(t)U(t)F_2^*(t), \quad X(t_0) = X_0,
\]

with output signal

\[
Y(t) = K_1(t)X(t)K_2^*(t)\]

and control \( U(t) \). First, we convert the system into a corresponding Kronecker product system with the help of Kronecker product technique, and its general solution is presented in terms of two transition matrices of the systems \( X'(t) = A(t)X(t) \) and \( X'(t) = B^*(t)X(t) \). Then, a set of necessary and sufficient conditions are presented for the complete controllability and complete observability of the \( \Delta \)-differential Kronecker product system.

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1. Introduction
The importance of matrix Sylvester systems on time scales is an interesting area of current research. Which arise in number of areas of control engineering problems, dynamical systems, and feedback systems are well known. There are many results from differential equations that carry over quite naturally and easily to difference equations, while others have a completely different structure for their continuous counterparts. The study of Sylvester system on time scales sheds new light on the discrepancies between continuous and discrete Sylvester systems. It is also prevents one from proving a result twice, once for continuous and once for discrete systems. The general idea, which is the main goal of Bhoner and Peterson’s introductory text [1] is to prove a result for a first order differential equation when the domain of the unknown function is so-called timescale. The two main objectives of this paper are therefore (1) to develop the theory and methods to solve Sylvester system on time scales (2) to explore the techniques of controllability and observability.

In this paper, we focus our attention to study of \( \Delta \)-differentiable matrix Sylvester system

\[
X'(t) = A(t)X(t) + X(t)B(t) + F_1(t)U(t)F_2^*(t), \quad X(t_0) = X_0, \quad (1.1)
\]

\[
Y(t) = K_1(t)X(t)K_2^*(t) \quad (1.2)
\]

where \( X(t) \) is an \( n \times n \) matrix, \( U(t) \) is \( m \times n \) input matrix called control and \( Y(t) \) is \( r \times n \) output matrix. Here \( A(t), B(t), F_1(t) \) and \( K_i(t) \) are \( n \times n, n \times n, n \times m \) and \( r \times n \) matrices respectively. \( F_2(t), K_2(t) \) are matrices of order \( n \times n \). Many authors [4], [10] obtained controllability and observability criteria for similar systems of the type (1.1) and (1.2) with \( B(t) = 0, F_2(t) \) and \( K_2(t) \) are identity matrices. This paper is well organized as follows: Section 2 we study some basic properties of Kronecker product of matrices and develop preliminary results by converting the given problem into a Kronecker product problem. The solution to the corresponding initial value problem obtained in terms of two transition matrices of the systems \( X'(t) = A(t)X(t) \) and \( X'(t) = B^*(t)X(t) \) (*denotes the transpose of a matrix), by using the standard technique of variation of parameters [6]. Also here we present...
some basic results relating to time scales. Section 3 we address the necessary and sufficient conditions for complete controllability and complete observability under certain smoothness conditions.

2. Preliminaries

In this section we present some basic definitions, notations and results which are useful for later discussion.

Definition 2.1. [3] If \( P, Q \in \mathbb{C}^{n \times n} \) are two square matrices of order ‘n’ then their Kronecker product(or direct product or tensor product) is denoted by \( P \otimes Q \in \mathbb{C}^{n^2 \times n^2} \) is defined to be partition matrix

\[
P \otimes Q = \begin{bmatrix}
p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\
p_{21}Q & p_{22}Q & \cdots & p_{2n}Q \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1}Q & p_{n2}Q & \cdots & p_{nn}Q
\end{bmatrix}
\]

We shall make use of vector valued function denoted by \( \text{Vec} \) of a matrix \( P = \{p_{ij}\} \in \mathbb{C}^{n \times n} \) defined by

\[
\hat{P} = \text{Vec} P = \begin{bmatrix}
p_{1j} \\
p_{2j} \\
\vdots \\
p_{nj}
\end{bmatrix}
\]

where \( 1 \leq j \leq n \)

it is clear that \( \text{Vec} P \) is of order \( n^2 \).

The Kronecker product has the following properties[3]

1. \( (P \otimes Q)^* = P^* \otimes Q^* \) (\( P^* \) denotes the transpose of \( P) \)
2. \( (P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1} \)
3. The mixed product rule \( (P \otimes Q)((M \otimes N) = (PM \otimes QN) \).This rule holds good, provided the dimension of the matrices are such that the various expressions exist.
4. If \( P(t) \) and \( Q(t) \) are matrices, then \( (P \otimes Q)' = P' \otimes Q' \) \( (P' = dP/dt) \)
5. \( \text{Vec}(PYQ) = (Q \otimes P)\text{Vec} \ Y \)
6. If \( P \) and \( Q \) are matrices both of order \( n \times n \) then

   (i) \( \text{Vec}(PX) = (I_n \otimes P)\text{Vec} X \)  
   (ii) \( \text{Vec}(XP) = (P \otimes I_n)\text{Vec} X \)

Now we introduce some basic definitions and results on time scales \( T[1][5] \) needed in our subsequent discussion.

A Timescale \( T \) is a closed subset of \( \mathbb{R} \); and examples of time scales include \( \mathbb{N}; \mathbb{Z}; \mathbb{R}, \) Fuzzy sets etc. The set \( Q = \{t \in \mathbb{R} / Q, 0 \leq t \leq 1\} \) are not time scales. Time scales need not necessarily be connected. In order to overcome this deficiency, we introduce the notion of jump
operators. Forward (backward) jump operator $\sigma(t)$ or $t < \sup T$ (respectively $\rho(t)$ at $t$ for $t > \inf T$) is given by $\sigma(t) = \inf\{s \in T : s > t\}$, $\rho(t) = \sup\{s \in T : s < t\}$, for all $t \in T$. The graininess function $\mu : T \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. Throughout we assume that $T$ has a topology that it inherits from the standard topology on the real number $R$. The jump operators $\sigma$ and $\rho$ allow the classification of points in a time scale in the way: If $\sigma(\tau) > \tau$, then the point $\tau$ is called right scattered; while if $\rho(\tau) < \tau$, then $\tau$ is termed left scattered. If $t < \sup T$ and $\sigma(t) = t$, then the point ' $t$' is called right dense: while if $t > \inf T$ and $\rho(t) = t$, then we say ' $t$' is left-dense. We say that $f : T \rightarrow R$ is rd-continuous provided $f$ is continuous at each right-dense point of $T$ and will be denoted by $C_{rd}$.

A function $F : T \rightarrow T$ is said to be differentiable in $t \in T^k = \{T \setminus (\rho(t) \max(T), \max t)\}$ if $\lim_{\sigma(t) \rightarrow x} \frac{f(\sigma(t) - f(s))}{\sigma(t) - s}$ where $s \in T-\{\sigma(t)\}$ exist and is said to be differentiable on $T$ provided it is differentiable for each $t \in T^k$. A function $F : T \rightarrow T$, with $F(t) = \tilde{f}(t)$ for all $t \in T^k$ is said to be integrable, if $\int_{\sigma}^{\tau} f(\tau) \Delta \tau = \dot{F}(\tau) - \dot{F}(s)$ where $\dot{F}$ is anti derivative of $f$ and for all $s, t \in T$. Let $f : T \rightarrow T$, and if $T = R$ and $a, b \in T$, then $f^a(t) = f(t)$ and $\int_{a}^{b} f(t) dt = \int_{a}^{b} f(t) \Delta t$.

If $T = Z$, then $f^A(t) = \Delta f(t) = f(t+1) - f(t)$ and $\int_{a}^{b} f(t) \Delta t = \sum_{k=a}^{b-1} f(k)$ if $a < b$

If $f, g : T \rightarrow X (X$ is a Banach space) be differentiable in $t \in T^k$. Then for any two scalars $\alpha, \beta$ the mapping $\alpha f + \beta g$ is differentiable in $t$ and further we have:

1. $(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t)$
2. $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)$
3. $(f(\sigma(t))^\Delta(t) = f(t) + \mu(t)f^\Delta(t)$
4. $(kf)^\Delta(t) = k f^\Delta(t)$, for any scalar $k$.

If $f$ is $A$-differentiable, then $f$ is continuous. Also if $t$ is right scattered and $f$ is continuous at $t$ then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$ 

Now by applying the Vec operator to the $A$-differentiable matrix Sylvester’s system (1.1) also the output equation (1.2) and using Kronecker product properties, we have

$$\Psi^\Delta(t) = G(t)\Psi(t) + \left[F_2 \otimes F_1\right]\dot{U}(t); \quad \Psi(t_0) = \Psi_0; \quad (2.1)$$
$$\dot{Y}(t) = [K_2 \otimes K_1]\dot{\psi}(t) \quad (2.2)$$

where $\Psi(t) = \text{Vec} X(t)$, $\dot{U}(t) = \text{Vec} U(t)$, $\dot{Y}(t) = \text{Vec} Y(t)$ and $G(t) = [B^T \otimes I + I \otimes A]$, is a $n^2 \times n^2$ matrix. Let $A(t)$ and $B(t)$ be regressive and rd-continuous. From the definition of Kronecker product $G : T^k \rightarrow R^{n^2}$ is regressive and rd-continuous.

When $t = R$, (2.1) is equivalent to

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\(\Psi'(t) = G(t)\Psi(t) + [F_2 \otimes F_1](t)\hat{U}(t); \quad \Psi(t_0) = \Psi_0\)

and when \(t = Z\), (2.1) is equivalent to

\(\Delta\Psi(n) = G(n)\Psi(n) + [F_2 \otimes F_1](n)\hat{U}(n); \quad \Psi(n_0) = \Psi_0\)

System (2.1) and (2.2) is called the Kronecker product system associated with (1.1) and (1.2).

It is easily seen that, if \(X(t)\) is the solution of (1.1) then \(\text{Vec}(X(t)) = \Psi(t)\) is the solution of (2.1) and vice-versa.

Now we confine our attention to corresponding homogeneous matrix system of (2.1) given by

\[\Psi^\Delta(t) = G(t)\Psi(t)\]  

(2.3)

**Lemma 2.1.** Let \(\phi_1(t,s)\) and \(\phi_2(t,s)\) denote state transition matrices of the systems \(X^\Delta(t) = A(t)X(t)\) and \(X^\Delta(t) = B^*(t)X(t)\) respectively. Then the matrix \(\phi(t,s)\) defined by

\[\phi(t,s) = \phi_1(t,s) \otimes \phi_2(t,s)\]  

(2.4)

is the state transition matrix of (2.3) and every solution of (2.3) is of the form \(\Psi(t) = \phi(t,s)C\) (where \(C\) is any constant vector of order \(n^2\)).

**Proof.** Consider

\[\phi^\Delta(t,s) = \phi_1^\Delta(t,s) \otimes \phi_2(t,s) + \phi_2(t,s) \otimes \phi_1^\Delta(t,s) = [B^* \otimes I_n + I_n \otimes A]\phi_1(t,s) \otimes \phi_2(t,s) = \phi(t,s)\]

also

\[\phi(t,t) = \phi_1(t,t) \otimes \phi_2(t,t) = I_n \otimes I_n = I_n;\]

hence \(\phi(t,s)\) is the transition matrix of (2.3). Moreover it can be easily seen that \(\Psi(t)\) is a solution of (2.3) and every solution of (2.3) is of this form.

**Theorem 2.1.** [6] Let \(\phi(t,s) = \phi_2(t,s) \otimes \phi_1(t,s)\) be a transition matrix (2.3), then the unique solution of (2.1), subject to the initial condition \(\Psi(t_0) = \Psi_0\) is

\[\Psi(t) = \phi(t,t_0)[\Psi_0 + \int_{t_0}^{t}\phi(t_0,s)(F_2 \otimes F_1)\hat{U}(s)\Delta s].\]

3. Controllability and Observability of \(\Delta\)-differential systems

In this section, we prove necessary and sufficient conditions for controllability and observability of the system (2.1) and (2.2).

**Definition 3.1.** The \(\Delta\)-differential systems \(S_1\) given by (2.1) and (2.2) is said to be completely controllable if for \(t_0\), any initial state \(\Psi(t_0) = \Psi_0\) and any given final state \(\Psi_f\), there exists a finite time \(t_1 > t_0\) and a control \(\hat{U}(t), t_0 \leq t \leq t_1\) such that \(\Psi(t_1) = \Psi_f\).

**Theorem 3.1.** The time scale dynamical system \(S_1\) is completely controllable on the closed interval \(J = [t_0, t_1]\) if and only if the \(n^2 \times n^2\) symmetric controllability matrix

\[V(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, s)(F_2 \otimes F_1)\phi^*(t_0, s)\Delta s\]

(3.1)

where \(\phi(t,s)\) is defined in (2.4), is nonsingular. In this case the control

\[\hat{U}(t) = -(F_2 \otimes F_1)^*(t)\phi^*(t_0, s)V^{-1}(t_0, t_1)[\Psi_0 - \phi(t_0, t_1)\Psi_f]\]

(3.2)

defined on \(t_0 \leq t \leq t_1\), transfers \(\Psi(t_0) = \Psi_0\) to \(\Psi(t_1) = \Psi_f\).

**Proof.** Suppose that \(V(t_0, t_1)\) is nonsingular, then the control defined by (3.2) exists. Now substituting (3.2) in (2.5) with \(t = t_1\), we have
\[
\Psi(t_1) = \phi(t_1, t_0)\left[\Psi_0 - \int_{t_0}^{t_1} \phi(t_0, \sigma(s))(F_2 \otimes F_1)(s)(F_2 \otimes F_1)^*(s)\phi^*(t_0, \sigma(s))V^{-1}(t_0, t_1)\{\Psi_0 - \phi(t_0, t_1)\Psi_f}\Delta s\right]
\]

\[
= \phi(t_1, t_0)\phi(t_0, t_1) = \Psi_f
\]
hence the dynamical system S1 is completely controllable.

Next suppose that the dynamical system S1 is completely controllable on \(J\), then we have to show that \(V(t_0, t_1)\) is nonsingular. Then there exists a non zero \(n^2 \times 1\) vector \(\alpha\) such that

\[
\alpha^*V(t_0, t_1)\alpha = \int_{t_0}^{t_1} \alpha^*\phi(t_0, \sigma(s))(F_2 \otimes F_1)(s)(F_2 \otimes F_1)^*(s)\phi^*(t_0, \sigma(s))\alpha\Delta s
\]

\[
= \int_{t_0}^{t_1} \theta^*(\sigma(s), t_0)\theta(\sigma(s), t_0)\Delta s = \int_{t_0}^{t_1} \|\theta\|^2 \Delta s \geq 0 \quad (3.3)
\]

where \(\theta = (F_2 \otimes F_1)^*(s)\phi^*(t_0, \sigma(s))\alpha\). From (3.3) \(V(t_0, t_1)\) is positive semi definite.

Suppose that there exists some \(\beta \neq 0\) such that \(\beta^*V(t_0, t_1)\beta = 0\) then from (3.3) with \(\theta = \eta\) when \(\alpha = \beta\), implies

\[
\int_{t_0}^{t_1} \|\eta\|^2 \Delta s = 0
\]

using the properties of norms, we have

\[
\eta(\sigma(s), t_0) = 0, t_0 \leq t \leq t_1 \quad (3.4)
\]

since S1 is completely controllable, so there exists a control \(\hat{U}(t)\) making

\[
\Psi(t_1) = 0 \quad \text{if} \quad \Psi(t_0) = \beta . \quad \text{Hence from (2.5) we have}
\]

\[
\beta = -\int_{t_0}^{t_1} \phi(t_0, \sigma(s))(F_2 \otimes F_1)(s)\hat{U}(s)\Delta s
\]

Consider

\[
\|\beta\|^2 = \beta^*\beta = \int_{t_0}^{t_1} \hat{U}^*(s)(F_2 \otimes F_1)^*(s)\phi^*(t_0, \sigma(s))\beta\Delta s
\]

\[
= -\int_{t_0}^{t_1} \hat{U}^*(s)\eta(\sigma(s), t_0)\Delta s = 0
\]

hence \(\beta = 0\), which is a contradiction to our assumption. Thus \(V(t_0, t_1)\) is positive definite and is therefore non singular.

We now turn our attention to the concept of observability on a timescale dynamical system.

**Definition 3.2.** The timescale dynamical system (2.1) is completely observable on \(J = [t_0, t_1]\) if for any time \(t_0\) and any initial state \(\Psi(t_0) = \Psi_0\) there exists a finite time \(t_1 > t_0\) such that the knowledge of \(\hat{U}(t)\) and \(\hat{Y}(t)\) for \(t_0 \leq t \leq t_1\) suffices to determine \(\Psi(t_0) = \Psi_0\) uniquely.

Now we present a necessary and sufficient condition for the system (2.1) to be completely observable.

**Theorem 3.2.** The system S1 is completely observable on \(J\) if and only if the \(n^2 \times n^2\) symmetric observability matrix

\[
L(t_0, t_1) = \int_{t_0}^{t_1} \phi^*(s, t_0)(K_2 \otimes K_1)^*(s)(K_2 \otimes K_1)(s)\phi(s, t_0)\Delta s
\]

is non singular.

**Proof.** Suppose that \(L(t_0, t_1)\) is non singular. It is simpler to consider the case of zero input, and it does not entail any loss of generality. Since the concept is not altered in the presence of...
a known input signal. Implies \( \hat{Y}(t) = [K_2 \otimes K_1] \psi(t) \) since from \( \Psi(t) = \phi(t, t_0)\Psi_0 \)

we have

\[
\hat{Y}(t) = [K_2 \otimes K_1] \phi(t, t_0)\Psi_0
\]

(3.5)

multiplying (3.5) on the left by \( \phi^*(t, t_0)(K_2 \otimes K_1)^*(t) \) and integrating from \( t_0 \) to \( t_1 \) we obtain

\[
\int_{t_0}^{t_1} \phi^*(s, t_0)(K_2 \otimes K_1)^*(s)\hat{Y}(s)ds = L(t_0, t_1)\Psi_0
\]

since \( L(t_0, t_1) \) is non singular, \( \Psi_0 \) can be determined uniquely. Hence the dynamical system S1 is completely observable.

Conversely suppose that the dynamical system S1 is completely observable. Then we prove that \( \Psi_0 \) is non singular. Since \( L(t_0, t_1) \) is symmetric, we can construct the quadratic form

\[
\alpha^*L(t_0, t_1)\alpha = \int_{t_0}^{t_1} \|\eta(s, t_0)\|^2ds \geq 0
\]

where \( \alpha \) is an arbitrary column \( n^2 \)-vector and \( \eta(s, t_0) = (K_2 \otimes K_1)(s)\phi(t_0, \sigma(s))\alpha \). From (3.6) \( L(t_0, t_1) \) is positive semi definite. Suppose that there exists some \( \beta \neq 0 \) such that \( \beta^*L(t_0, t_1)\beta = 0 \) then from (3.6) with \( \eta=0 \) when \( \alpha=\beta \), implies

\[
\int_{t_0}^{t_1} \|\theta(s, t_0)\|^2ds = 0 \Rightarrow \theta(s, t_0) = 0, t_0 \leq s \leq t_1.
\]

From (3.5), this implies that when \( \Psi_0=\beta \), the ou put is identically zero throughout the interval, so that \( \Psi_0 \) can not be determined in this case from knowledge of \( \hat{Y}(t) \). This contradicts the supposition that S1 is completely observable.

Hence \( L(t_0, t_1) \) is positive definite, therefore non –singular. The proof is complete.

References